A second-order implicit difference method for time-space advection-diffusion equation

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Outline

Outline

Application

The fractional advection-diffusion equation (ADE) is a more suitable model for many problems, such as continuous time random walk (Wang et al. 11'), porous media transmission(多孔介质传输) (Benson et al. 01'), entropy(熵) (Povstenko et al. 15'), hydrology(水文地理学) (Benson et al. 00'), Brownian motion(布朗运动) (Benson et al. 00'), physics (Sokolov et al. 02') and image processing (Bai et al. 07').

Outline

Pioneering work

Last years, the work about numerically handling ADE is not too much and can be summarized as:

- `ms-` TADE with convergence order $\mathcal{O}(\tau^{2-\alpha} + h)$ (Lin et al. 07'),

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- ms- TSADE with convergence order $\mathcal{O}(\tau + h)$ (Qin & Zhang 08', Zhang 09', Shao & Ma 16'),
- ms- TSADE with convergence order $\mathcal{O}(\tau^{2-\alpha} + h)$ (Liu et al. 12', Zhao 16'),

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- `ms-` TSADE with convergence order $\mathcal{O}(\tau^{2-\alpha} + h)$ (Liu et al. 12', Zhao 16'),
- `ms-` TSADE with convergence order $\mathcal{O}(\tau^2 + h^2)$, but didn't consider the general situation (Gu et al. in pressed).
- ...

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Outline

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2. An implicit difference scheme for TSADE
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   - The discretization of space
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Motivation

Construct an implicit difference scheme with convergent $\mathcal{O}(\tau^2 + h^2)$, to solve the time-space fractional advection-diffusion equation (TSADE):

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$$\begin{cases} D_{0,t}^{\theta} u(x,t) = - d_+(t) D_{0,x}^{\alpha} u(x,t) - d_-(t) D_{x,L}^{\alpha} u(x,t) \\ \qquad\qquad + e_+(t) D_{0,x}^{\beta} u(x,t) + e_-(t) D_{x,L}^{\beta} u(x,t) + f(x,t), \\ u(x,0) = u_0(x), \qquad 0 \le x \le L, \\ u(0,t) = u(L,t) = 0, \qquad 0 \le t \le T, \end{cases} \tag{1}$$

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where $\theta, \alpha \in (0,1]$, $\beta \in (1,2]$.

Outline

Definition

The time fractional derivative in (1) is the Caputo fractional derivative of order θ denoted by

$$D_{0,t}^{\theta}u(x,t) = \frac{1}{\Gamma(1-\theta)}\int_0^t \frac{\partial u(x,\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^{\theta}}$$

Definition

The left and right space fractional derivatives in (1) are the Riemann-Liouville fractional derivatives which are defined as follows.

1. left Riemann-Liouville fractional derivative:

$$D_{0,x}^{\gamma}u(x,t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_0^x\frac{u(\eta,t)}{(x-\eta)^{\gamma-n+1}}d\eta,\ \ n-1\leq\gamma<n$$

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2. right Riemann-Liouville fractional derivative:

$$D_{x,L}^{\gamma}u(x,t)=\frac{(-1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_x^L\frac{u(\eta,t)}{(\eta-x)^{\gamma-n+1}}d\eta,\ \ n-1\leq\gamma<n$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Outline

$L2-1_\sigma$ difference formula

Before derive the proposed scheme, we first introduce the mesh $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{x_i = ih, i = 0, 1, \cdots, N; x_0 = 0, x_N = L\}$, $\bar{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \cdots, M; t_M = T\}$. Then, we employ Alikhanov's formula to discrete time at fixed point $t_{j+\sigma}(\sigma = 1 - \frac{\theta}{2})$:

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$$
\begin{aligned}
D_{0,t}^{\theta} u(x_i, t_{j+\sigma}) &= \frac{\tau^{1-\theta}}{\Gamma(2-\theta)} \sum_{s=0}^{j} c_{j-s}^{(\theta,\sigma)} u_{t,s} + \mathcal{O}(\tau^{3-\theta}) \\
&= \Delta_{0t_{j+\sigma}}^{\theta} u + \mathcal{O}(\tau^{3-\theta}),
\end{aligned}
$$

where $u_{t,s} = (u(x_i, t_{s+1}) - u(x_i, t_s))/\tau$.

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$$D_{x,L}^{\gamma} u(x_i, t_{j+\sigma}) = \frac{1}{h^{\alpha}} \sum_{k=0}^{N-i+1} \omega_k^{(\gamma)} u(x_{i+k-1}, t_{j+\sigma}) + \mathcal{O}(h^2),$$

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where

$$\omega_0^{(\gamma)} = \frac{\gamma}{2} g_0^{(\gamma)}, \quad \omega_k^{(\gamma)} = \frac{\gamma}{2} g_k^{(\gamma)} + \frac{2-\gamma}{2} g_{k-1}^{(\gamma)}, \ k \geq 1$$

Outline

The implicit scheme

For simplicity, let us introduce some notations

$$u(x_i,t_j)\approx u_i^j,\ \ u_i^{(\sigma)}=\sigma u_i^{j+1}+(1-\sigma)u_i^j,$$
$$d_{\pm}^{j+\sigma}=d_{\pm}(t_{j+\sigma}),\ \ e_{\pm}^{j+\sigma}=e_{\pm}(t_{j+\sigma}),\ \ f_i^{j+\sigma}=f(x_i,t_{j+\sigma}),$$
$$\delta_{x,+}^{\gamma}u_i^j=\frac{1}{h^{\gamma}}\sum_{k=0}^{i+1}\omega_k^{(\gamma)}u_{i-k+1}^j,\delta_{x,-}^{\gamma}u_i^j=\frac{1}{h^{\gamma}}\sum_{k=0}^{N-i+1}\omega_k^{(\gamma)}u_{i+k-1}^j,$$

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$$\begin{aligned}\delta_h^{\alpha,\beta}u_i^{(\sigma)}=&-d_+^{j+\sigma}\delta_{x,+}^{\alpha}u_i^{(\sigma)}-d_-^{j+\sigma}\delta_{x,-}^{\alpha}u_i^{(\sigma)}\\&+e_+^{j+\sigma}\delta_{x,+}^{\beta}u_i^{(\sigma)}+e_-^{j+\sigma}\delta_{x,-}^{\beta}u_i^{(\sigma)}.\end{aligned}$$

The implicit scheme

With these notations, we shall see that the solution of (1) can be approximated by the following implicit difference scheme for $(x, t) = (x_i, t_{j+\sigma}) \in \bar{\omega}_{h\tau}$, $i = 1, 2, \ldots, N-1$, $j = 0, 1, \ldots, M-1$:

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$$\Delta^{\theta}_{0,t_{j+\sigma}} u_i = \delta^{\alpha,\beta}_h u^{(\sigma)}_i + f^{j+\sigma}_i.$$

The implicit scheme

Then we derive the implicit difference schemes with the approximation order $\mathcal{O}(\tau^2 + h^2)$:

$$\begin{cases} \Delta^{\theta}_{0,t_{j+\sigma}} u_i = \delta_h^{\alpha,\beta} u_i^{(\sigma)} + f_i^{j+\sigma}, & 1 \le i \le N-1, \quad 0 \le j \le M-1, \\ u_i^0 = u_0(x_i), & 1 \le i \le N-1, \\ u_0^j = u_N^j = 0, & 0 \le j \le M-1. \end{cases} \tag{2}$$

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It is interesting to note that for $\theta \to 1$ we obtain the Crank-Nicolson difference scheme.

Stability and convergence

Theorem

Denote $u^{j+1} = (u_1^{j+1}, u_2^{j+1}, \ldots, u_{N-1}^{j+1})^T$ *and* $\left\| f^{j+\sigma} \right\|^2 = h\sum_{i=1}^{N-1} f^2(x_i, t_{j+\sigma})$. *Then the implicit difference scheme (2) is unconditionally stable for* $N \geq 3$, *and the following a priori estimate holds:*

$$\left\| u^{j+1} \right\|^2 \leq \left\| u^0 \right\|^2 + \frac{T^\theta \Gamma(1-\theta)}{c \ln 2} \left\| f^{j+\sigma} \right\|^2, \quad 0 \leq j \leq M-1.$$

Stability and convergence

Theorem

Suppose that $u(x,t)$ is the solution of (1) and $\{u_i^j \mid x_i \in \bar{\omega}_h, \ 0 \le j \le M\}$, is the solution of the implicit difference scheme (2). Denote $\xi_i^j = u(x_i, t_j) - u_i^j, \ x_i \in \bar{\omega}_h, \ 0 \le j \le M$. Then there exists a positive constant $\tilde{c}$ such that

$$\|\xi^j\| \le \tilde{c}(\tau^2 + h^2), \quad 0 \le j \le M.$$

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Numerical results

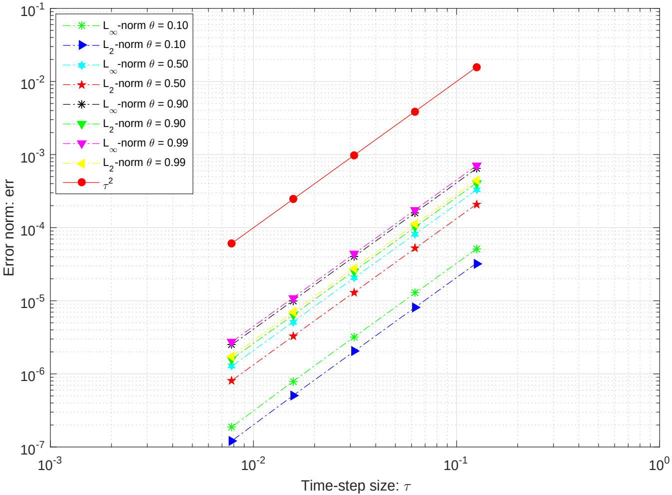
Example 1. In this example, we consider the equation (1) on space interval $[0, L] = [0, 1]$ and time interval $[0, T] = [0, 1]$ with advection coefficients $d_+(t) = d_+$, $d_-(t) = d_-$, diffusion coefficients $e_+(t) = e_+$, $e_-(t) = e_-$, initial condition $u(x, 0) = 0$, and the exact solution is $u(x, t) = t^{2+\theta} x^2 (1 - x)^2$.

Numerical results

Table: L_2-norm and maximum norm error behavior versus τ-grid size reduction when $\alpha = 0.6$, $\beta = 1.8$, $d_+ = 10$, $d_- = 11$, $e_+ = 1e-11$, $e_- = 1e-12$ and $h = 1/3000$ in Example 1.

θ	τ	$\|\xi\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	CO in $\|\cdot\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	$\max_{0\le n\le M}\|\xi^n\|_0$	CO in $\|\cdot\|_0$
0.10	1/8	5.1122e-05	–	3.2467e-05	–
	1/16	1.2793e-05	1.9986	8.1264e-06	1.9983
	1/32	3.1906e-06	2.0035	2.0283e-06	2.0024
	1/64	7.8749e-07	2.0185	5.0230e-07	2.0136
	1/128	1.8710e-07	2.0734	1.2096e-07	2.0541
0.50	1/8	3.2725e-04	–	2.0791e-04	–
	1/16	8.2154e-05	1.9940	5.2193e-05	1.9940
	1/32	2.0568e-05	1.9979	1.3068e-05	1.9978
	1/64	5.1359e-06	2.0017	3.2645e-06	2.0011
	1/128	1.2740e-06	2.0112	8.1142e-07	2.0083
0.90	1/8	6.4147e-04	–	4.0767e-04	–
	1/16	1.6053e-04	1.9985	1.0202e-04	1.9986
	1/32	4.0133e-05	2.0000	2.5504e-05	2.0000
	1/64	1.0021e-05	2.0017	6.3696e-06	2.0015
	1/128	2.4943e-06	2.0063	1.5869e-06	2.0050
0.99	1/8	7.0327e-04	–	4.4702e-04	–
	1/16	1.7582e-04	1.9999	1.1176e-04	1.9999
	1/32	4.3946e-05	2.0003	2.7935e-05	2.0003
	1/64	1.0976e-05	2.0014	6.9786e-06	2.0011
	1/128	2.7340e-06	2.0053	1.7398e-06	2.0040

Numerical results

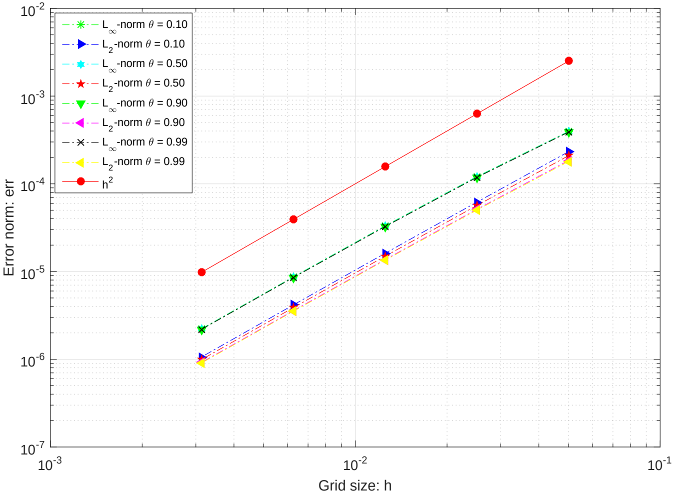


Numerical results

Table: L_2-norm and maximum norm error behavior versus grid size reduction when $\alpha = 0.6$, $\beta = 1.8$, $d_+ = 10$, $d_- = 11$, $e_+ = 1e-11$, $e_- = 1e-12$ and $\tau = h$ in Example 1.

θ	τ	$\|\xi\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	CO in $\|\cdot\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	$\max_{0\le n\le M}\|\xi^n\|_0$	CO in $\|\cdot\|_0$
0.10	1/20	3.9523e-04	–	2.3180e-04	–
	1/40	1.2051e-04	1.7135	6.1600e-05	1.9119
	1/80	3.3072e-05	1.8655	1.6086e-05	1.9372
	1/160	8.6598e-06	1.9332	4.1448e-06	1.9564
	1/320	2.2169e-06	1.9658	1.0576e-06	1.9705
0.50	1/20	3.9330e-04	–	2.0870e-04	–
	1/40	1.1971e-04	1.7161	5.6407e-05	1.8875
	1/80	3.2868e-05	1.8648	1.4879e-05	1.9226
	1/160	8.6136e-06	1.9320	3.8579e-06	1.9474
	1/320	2.2069e-06	1.9646	9.8825e-07	1.9649
0.90	1/20	3.8941e-04	–	1.8407e-04	–
	1/40	1.1832e-04	1.7186	5.0980e-05	1.8522
	1/80	3.2522e-05	1.8632	1.3632e-05	1.9030
	1/160	8.5355e-06	1.9299	3.5627e-06	1.9359
	1/320	2.1899e-06	1.9626	9.1705e-07	1.9579
0.99	1/20	3.8810e-04	–	1.7924e-04	–
	1/40	1.1789e-04	1.7189	4.9922e-05	1.8441
	1/80	3.2418e-05	1.8626	1.3388e-05	1.8988
	1/160	8.5121e-06	1.9292	3.5048e-06	1.9335
	1/320	2.1849e-06	1.9620	9.0304e-07	1.9565

Numerical results



Numerical results

Example 2. In this example, the left-sided and right-sided advection-diffusion coefficients are given by

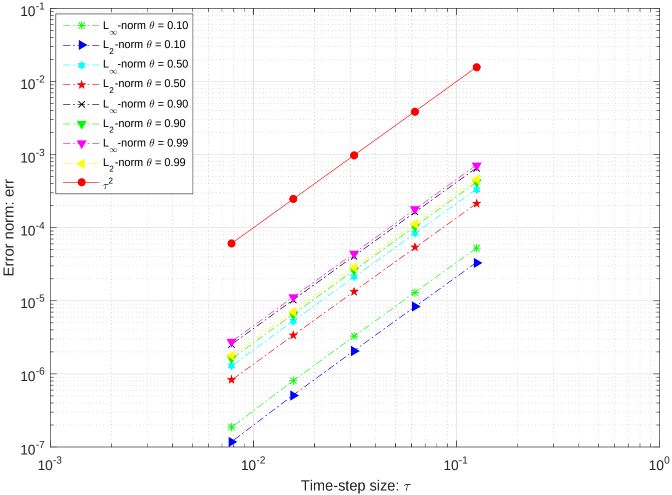
$$d_+(t) = e^t, \qquad d_-(t) = 3e^{-t},$$
$$e_+(t) = (1+t)^2, \qquad e_-(t) = 1 + t^2.$$

Numerical results

Table: L_2-norm and maximum norm error behavior versus τ-grid size reduction when $\alpha = 0.6$, $\beta = 1.8$, and $h = 1/3000$ in Example 2.

θ	τ	$\|\xi\|_{C(\bar{\omega}_{h\tau})}$	CO in $\|\cdot\|_{C(\bar{\omega}_{h\tau})}$	$\max_{0\le n\le M}\|\xi^n\|_0$	CO in $\|\cdot\|_0$
0.10	1/8	5.2321e-05	–	3.3293e-05	–
	1/16	1.3096e-05	1.9982	8.3340e-06	1.9981
	1/32	3.2603e-06	2.0061	2.0748e-06	2.0061
	1/64	7.9868e-07	2.0293	5.0828e-07	2.0292
	1/128	1.8626e-07	2.1003	1.1850e-07	2.1007
0.50	1/8	3.3405e-04	–	2.1261e-04	–
	1/16	8.3929e-05	1.9928	5.3418e-05	1.9928
	1/32	2.1014e-05	1.9978	1.3374e-05	1.9979
	1/64	5.2426e-06	2.0030	3.3366e-06	2.0030
	1/128	1.2947e-06	2.0177	8.2404e-07	2.0176
0.90	1/8	6.5352e-04	–	4.1601e-04	–
	1/16	1.6365e-04	1.9976	1.0417e-04	1.9976
	1/32	4.0921e-05	1.9997	2.6048e-05	1.9997
	1/64	1.0216e-05	2.0020	6.5027e-06	2.0021
	1/128	2.5376e-06	2.0093	1.6153e-06	2.0093
0.99	1/8	7.1584e-04	–	4.5567e-04	–
	1/16	1.7899e-04	1.9998	1.1395e-04	1.9996
	1/32	4.4733e-05	2.0004	2.8479e-05	2.0005
	1/64	1.1167e-05	2.0021	7.1095e-06	2.0021
	1/128	2.7756e-06	2.0084	1.7671e-06	2.0084

Numerical results



Numerical results

Table: L_2-norm and maximum norm error behavior versus grid size reduction when $\alpha = 0.6$, $\beta = 1.8$, and $\tau = h$ in Example 2.

θ	τ	$\|\xi\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	CO in $\|\cdot\|_{\mathcal{C}(\bar{\omega}_{h\tau})}$	$\max_{0\le n\le M}\|\xi^n\|_0$	CO in $\|\cdot\|_0$
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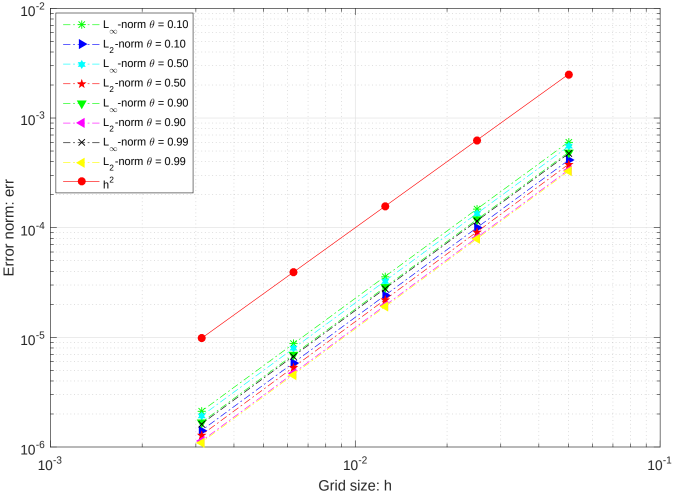
	1/40	1.4719e-04	2.0351	9.9593e-05	2.0471
	1/80	3.5781e-05	2.0404	2.4068e-05	2.0489
	1/160	8.6941e-06	2.0411	5.8187e-06	2.0483
	1/320	2.1128e-06	2.0409	1.4087e-06	2.0464
0.50	1/20	5.5002e-04	–	3.7743e-04	–
	1/40	1.3407e-04	2.0365	9.1172e-05	2.0496
	1/80	3.2522e-05	2.0435	2.1985e-05	2.0520
	1/160	7.8829e-06	2.0446	5.3038e-06	2.0514
	1/320	1.9110e-06	2.0444	1.2815e-06	2.0492
0.90	1/20	4.8566e-04	–	3.3574e-04	–
	1/40	1.1823e-04	2.0384	8.0954e-05	2.0522
	1/80	2.8599e-05	2.0475	1.9472e-05	2.0557
	1/160	6.9105e-06	2.0491	4.6857e-06	2.0551
	1/320	1.6699e-06	2.0490	1.1296e-06	2.0524
0.99	1/20	4.7183e-04	–	3.2657e-04	–
	1/40	1.1479e-04	2.0392	7.8721e-05	2.0526
	1/80	2.7752e-05	2.0484	1.8925e-05	2.0564
	1/160	6.7015e-06	2.0500	4.5520e-06	2.0558
	1/320	1.6181e-06	2.0502	1.0969e-06	2.0530

Numerical results



Conclusion and Future work

Conclusions:

- The FFT method is developed for saving computation complexity;
- Three fast Krylov subspace methods are employed for saving time;
- Theoretical and numerical results show our conclusions.

Future work:

- Other cheap and efficient finite difference formulae should be investigated;
- Tempered FDEs;
- The fractional order linear systems.

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Q & A

Thank you for attention!